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AN APPROACH TO PREDICTION OF FREE CONVECTION IN NON-NEWTONIAN FLUIDS

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Abstract—An approach to prediction of free convection in non-linear-viscous and Newtonian fluids at high Prandtl and Schmidt numbers is presented in the form of coupled external and internal asymptotic expansions. The method is shown to simplify essentially the solution of the initial problem allowing similarity solutions in particular. The solutions for steady and unsteady free convection are found.

NOMENCLATURE

$x = \frac{x'}{L}$, $y = \frac{y'}{L} Gr^{\frac{1}{2(n+1)}}$, $y_1 = y Pr_i^{\frac{n}{3n+1}}$, $y_2 = y$,
dimensionless coordinates;

x', y', t', dimensional coordinates;

u, $= u'[\log \beta_T(T_0 - T_\infty)]^{-\frac{1}{2}}$,

v, $= v'[\log \beta_T(T_0 - T_\infty)]^{-\frac{1}{2}} Gr^{\frac{1}{2(n+1)}}$,

u_1, $= u Pr_i^{\frac{n+1}{3n+1}}$, $v_1 = v Pr_i^{\frac{2n+1}{3n+1}}$, $u_2 = u$, $v_2 = v$,
dimensionless velocities;

u', v', dimensional velocities;

L, characteristic length;

β_T, volume expansion coefficient;

Pr_T, $= \frac{\rho C_p}{\lambda} L^{\frac{1-n}{1+n}} \left(\frac{\rho}{k}\right)^{-\frac{2}{n+1}} [L\beta_T g(T_0 - T_\infty)]^{\frac{3(n-1)}{2(n+1)}}$,

Pr_c, $= \frac{1}{\mathscr{D}} \left(\frac{\rho}{k}\right)^{-\frac{2}{n+1}} L^{\frac{1-n}{1+n}} [L\beta_T g(T_0 - T_\infty)]^{\frac{3(n-1)}{2(n+1)}}$,
modified Prandtl and Schmidt numbers;

Gr_T, $= \left(\frac{\rho}{k}\right)^2 L^{n+2} [\beta_T g(T_0 - T_\infty)]^{2-n}$,

Gr_c, $= \left(\frac{\rho}{k}\right)^2 L^{n+2} [\beta_c g |C_0 - C_\infty|]^{2-n}$,
modified Grashof numbers;

k, consistency coefficient;

n, non-Newtonian flow behaviour index;

$\omega(\Theta_T)$, function for temperature-dependent consistency coefficient;

Θ_T, $= \frac{T - T_\infty}{T_0 - T_\infty}$, $\Theta_c = \frac{C - C_\infty}{C_0 - C_\infty}$,
dimensionless temperature and concentration;

T_0, T_∞, absolute temperatures at a wall and at $y \to \infty$;

C_0, C_∞, concentrations at a wall and at $y \to \infty$;

η_1, $= y_1 x^{-\frac{n-\gamma}{3n+1}} \left[\frac{2n+1+\gamma}{3n+1}\right]^{\frac{n}{3n+1}} \cdot M^{\frac{1}{3n+1}}$

$f_1(\eta_1)$, $= \Psi x^{-\frac{2n+1+\gamma}{3n+1}} \left[\frac{2n+1+\gamma}{3n+1}\right]^{\frac{2n+1}{3n+1}} M^{-\frac{1}{3n+1}}$,

t, $= t'L[\log \beta_T(T_0 - T_\infty)]^{-\frac{1}{2}}$, $Fo = t Pr_i^{-\frac{n+1}{3n+1}}$,
similarity variables.

RECENTLY the interest in the problems of free convection liquids has grown considerably because of wide use of fluids (both Newtonian and non-Newtonian) in chemical, food and construction engineering, in petroleum production and power engineering. In the present work a method of solution is proposed for free convection near bodies submerged into fluid. New solutions for power-law non-Newtonian fluids illustrate its efficiency and operation possibilities. Dimensionless equations of unsteady-state thermal-concentrational free convection in the boundary-layer approach with regard for the temperature-dependent consistency coefficient are of the form

$$\begin{gathered}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\
= \frac{\partial}{\partial y}\left[\omega(\Theta_T)\left|\frac{\partial u}{\partial y}\right|^{n-1} \frac{\partial u}{\partial y}\right] + (\Theta_T + A\Theta_c)Mx^{\gamma}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\
\frac{\partial \Theta_T}{\partial t} + u\frac{\partial \Theta_T}{\partial x} + v\frac{\partial \Theta_T}{\partial y} = \frac{1}{Pr_T}\frac{\partial^2 \Theta_T}{\partial y^2}, \\
\frac{\partial \Theta_c}{\partial t} + u\frac{\partial \Theta_c}{\partial x} + v\frac{\partial \Theta_c}{\partial y} = \frac{1}{Pr_c}\frac{\partial^2 \Theta_c}{\partial y^2}.
\end{gathered} \tag{1}$$

The boundary conditions

$$\begin{gathered}
u = 0,\ \Theta_T = 0,\ \Theta_c = 0 \quad \text{at} \quad t = 0 (y > 0); \\
u = 0,\ v = 0,\ \Theta_T = 1,\ \Theta_c = 1 \quad \text{at} \quad y = 0, \\
u \to 0,\ \Theta_T \to 0,\ \Theta_c \to 0 \quad \text{at} \quad y \to \infty (t > 0).
\end{gathered} \tag{2}$$

Here

$$A = \text{sign}(C_0 - C_\infty)\left(\frac{Gr_c}{Gr_T}\right)^{\frac{1}{2-n}}$$

and geometrical parameters are: $M = 1$, $\gamma = 0$ for a vertical plate, $M = \cos\varphi$, $\gamma = 0$ for a wedge; $M = 1$, $\gamma = 1$ for a plane critical point.

Solution of the set of equations (1) with boundary conditions (2) is a matter of great mathematical difficulty. In particular, formulation of (1)–(2) permits no similarity solutions. The solution of the problem is considerably simplified at high Prandtl (Pr_T) and Schmidt (Pr_c) numbers because thermal and diffusional boundary layers are much thinner than dynamic one. This exactly favours the application of the method of coupled asymptotic expansions. It means that the problem is presented in the form of two asymptotic (internal and external) expansions describing the process in one and the same range of independent variables, but converging to the exact solution in different parts of this range (internal expansions converge near the wall, external, far from it). The principle of coupling should be used to close the system of equations necessary for determining these expansions.

The set of equations (1) for the internal asymptotic expansion is written as

$$\begin{gathered}\frac{\partial}{\partial y_1}\left[\omega(\Theta_T)\left|\frac{\partial u_1}{\partial y_1}\right|^{n-1}\frac{\partial u_1}{\partial y_1}\right] + (\Theta_T + A\Theta_c)Mx^\gamma = 0,\\ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y_1} = 0,\\ \frac{\partial \Theta_T}{\partial Fo} + u_1\frac{\partial \Theta_T}{\partial x} + v_1\frac{\partial \Theta_T}{\partial y_1} = \frac{Pr_i}{Pr_T}\frac{\partial^2\Theta_T}{\partial y_1^2},\\ \frac{\partial \Theta_c}{\partial Fo} + u_1\frac{\partial \Theta_c}{\partial x} + v_1\frac{\partial \Theta_c}{\partial y_1} = \frac{Pr_i}{Pr_c}\frac{\partial^2\Theta_c}{\partial y_1^2},\end{gathered} \tag{3}$$

where

$$Pr_i = \min\{Pr_T, Pr_c\}.$$

The condition $u = 0$ at $y \to \infty$ for the internal asymptotic expansion should be substituted by the restriction of u_1, which, from physical considerations, results in the requirement that $\partial u_1/\partial y_1 \to 0$. Thus, the boundary conditions for system (3) is

$$\begin{gathered}\Theta_T = \Theta_c = 0 \quad \text{at} \quad Fo = 0(y_1 > 0),\\ u_1 = v_1 = 0, \ \Theta_T = \Theta_c = 1 \quad \text{at} \quad y_1 = 0\\ \frac{\partial u_1}{\partial y_1} \to 0, \ \Theta_T \to 0, \ \Theta_c \to 0 \quad \text{at} \quad y_1 \to \infty(Fo > 0).\end{gathered} \tag{4}$$

The requirement for $\Theta_T = \Theta_c = 0$ in the external asymptotic expansion results in a set of equations

$$\begin{gathered}\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + v_2\frac{\partial u_2}{\partial y_2} = \frac{\partial}{\partial y_2}\left[\left|\frac{\partial u_2}{\partial y_2}\right|^{n-1}\frac{\partial u_2}{\partial y_2}\right],\\ \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y_2} = 0.\end{gathered} \tag{5}$$

The condition of adhesion on a surface cannot be now realized. A new condition on a surface is determined from "the principle of limiting coupling" [1]. Hence the boundary conditions for the system are:

$$\begin{gathered}u_2 = 0 \quad \text{at} \quad t = 0(y_2 > 0);\\ u_2 = U, \ v_2 = 0 \quad \text{at} \quad y_2 = 0\\ u_2 \to 0 \quad \text{at} \quad y_2 \to \infty(t > 0)\end{gathered}$$

where

$$U = \lim_{y_1 \to \infty} u_1(Fo, x, y_1). \tag{6}$$

The statement of the problem (1), (2) in the form of internal (3), (4) and external (5), (6) asymptotic expansions simplifies considerably its solution without deleterious effect on quantitative and qualitative estimates of the investigation process.

In contrast to the initial system, equations (3), (4) in case of steady-state convection, in particular, allows similarity solutions described by ordinary differential equations

$$\begin{gathered}\frac{d}{d\eta_1}[\omega(\Theta_T)|f_1''|^{n-1}f_1''] + \Theta_T + A\Theta_c = 0,\\ \Theta_T'' + \frac{Pr_T}{Pr_i}(\text{sign} f_1')f_1\Theta_T' = 0,\\ \Theta_c'' + \frac{Pr_c}{Pr_i}(\text{sing} f_1')f_1\Theta_c' = 0\end{gathered} \tag{7}$$

with the boundary conditions

$$\begin{gathered}f_1 = 0, \ f_1' = 0, \ \Theta_T = 1, \ \Theta_c = 1 \quad \text{at} \quad \eta_1 = 0;\\ f_1'' \to 0, \ \Theta_T \to 0, \ \Theta_c \to 0 \quad \text{at} \quad \eta_1 \to \infty.\end{gathered} \tag{8}$$

Local surface heat- and mass-transfer coefficients are of the form

$$Nu_k = -\Theta_k'(0)\left(\frac{2n+1+\gamma}{3n+1}\right)^{\frac{n}{3n+1}} \times M^{\frac{1}{3n+1}} Gr_T^{\frac{1}{2(n+1)}} Pr_i^{\frac{n}{3n+1}} x^{\frac{\gamma-n}{3n+1}}.$$

The numerical solution is obtained in [2]. The internal asymptotic expansion allows characteristics of surface heat and mass transfer and friction as well as temperature and concentration profiles to be determined.

To obtain complete velocity profiles over the whole test region, the solution should be found for the external asymptotic expansion of the problem.

From the limiting coupling principle and the solution obtained for the internal asymptotic expansion it follows

$$U = f_1'(\infty)Pr_i^{1-\frac{n+1}{3n+1}} x^{\frac{n+1+2\gamma}{3n+1}} M^{\frac{2}{3n+1}}\left(\frac{3n+1}{2n+1+\gamma}\right)^{\frac{n+1}{3n+1}}. \tag{9}$$

The transition in (5), (6) to stream function and the requirement of constant conformal invariance of the set of equations obtained relative to linear one-parametric transformation group [3] yield with regard for (9) similarity variables

$$\begin{gathered}\eta_2 = C_1 y_2 x^{-\frac{n^2+2n-1-2\gamma(2-n)}{(3n+1)(n+1)}}\\ f_2(\eta_2) = C_2\Psi x^{-\frac{2[n(n+2)+\gamma(2n-1)]}{(3n+1)(n+1)}}\end{gathered} \tag{10}$$

Here

$$C_1 = \mathscr{D}^{\frac{2-n}{n+1}}, \quad C_2 = \mathscr{D}^{\frac{1-2n}{1+n}}$$

$$\mathscr{D} = f_1'(\infty)Pr_i^{\frac{n+1}{-3n+1}} M^{\frac{2}{3n+1}}\left(\frac{3n+1}{2n+1+\gamma}\right)^{\frac{n+1}{3n+1}}.$$

Variables (10) reduce (5), (6) to the nonlinear differential equation

$$n|f_2''|^{n-1}f_2''' - \frac{n+1+2\gamma}{3n+1}(f_2')^2 + \frac{2[n(n+2)+\gamma(2n-1)]}{(3n+1)(n+1)} f_2 f_2'' = 0 \quad (11)$$

with the boundary conditions

$$f_2(0) = 0, \quad f_2'(0) = 1, \quad f_2'(\infty) \to 0. \quad (12)$$

Some results of numerical solution of (11) and (12) are shown in Fig. 1. Increase in pseudoplasticity (decrease of the non-Newtonian flow behaviour index n) increases the velocity boundary-layer thickness (in similarity variables) (Fig. 1a). The geometrical parameter γ influences slightly velocity profiles, Fig. 1(b).

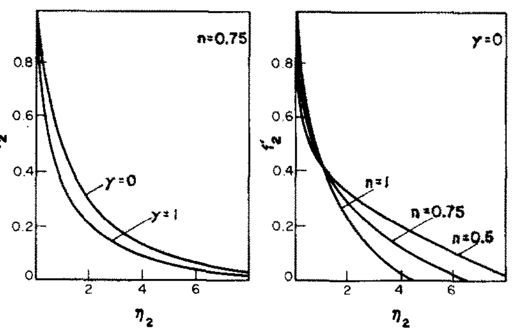

FIG. 1. Velocity profiles for external asymptotic expansion.

Temperature (concentration) and velocity profiles found by the method suggested and determined from the solution of complete problem (1), (2) with $n = 1$, $Pr_T = Pr_c = 100$, $A = 0$, $M = 1$, $\gamma = 0$ (for $n = 1$ problem (1)–(2) allows the similarity solutions) are presented in Fig. 2. The plots show fine agreement between the results. For example, the difference between exact and approximate heat- and mass-transfer and friction characteristics does not exceed 3.5%. The composite velocity profile includes the segments of the profiles for internal and external asymptotic expansions up to their intersection. With increasing Prandtl number, Pr_T, Schmidt number Pr_c, the accuracy of determining both the surfacial characteristics and temperatures, concentrations and velocity profiles increases.

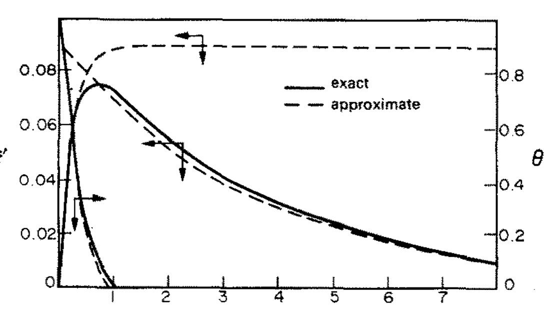

FIG. 2. Temperature and velocity profiles obtained from exact solution and by coupled asymptotic expansions.

It should be noted that the solution of (11)–(12) is universal relative to the Prandtl and Schmidt numbers, Grashof numbers (Gr_T, Gr_c) and function $\omega(\Theta_T)$.

Now consider the advantages of the method proposed. Firstly, it results in the problem allowing similarity solutions; secondly, only the ratio Pr_T/Pr_c remains in the equations instead of two independent parameters Pr_T, Pr_c. Also using the similarity variables, the problem for external asymptotic expansion is solved independently of that for internal expansion which considerably simplifies integration.

Consider the solution of unsteady-state problem (3)–(4) for thermal free convection near a vertical plate. The temperature-dependent consistency coefficient is neglected. The problem will be solved by a semi-integral finite-thickness boundary-layer approach [4].

Boundary conditions (4) are extended to $y_1 = \delta(Fo, x)$ and supplemented with the physical requirement

$$\frac{\partial\Theta}{\partial y_1} = 0 \quad \text{at} \quad y_1 = \delta(Fo, x). \quad (13)$$

It follows from motion equation (3) with regard for (13) and

$$\frac{\partial u_1}{\partial y_1} \geqslant 0, \quad \text{i.e.} \quad \left|\frac{\partial u_1}{\partial y_1}\right|^{n-1}\frac{\partial u_1}{\partial y_1} = \left(\frac{\partial u_1}{\partial y_1}\right)^n$$

inside the boundary layer, that

$$\frac{\partial u_1}{\partial y_1} = \left(\int_{y_1}^{\delta} \Theta \, dy\right)^{\frac{1}{n}}. \quad (14)$$

Integrating the energy equation from $y_1 = 0$ to $y_1 = \delta$ and allowing for the continuity equation and conditions (14), (4), we obtain the integral relationship

$$\frac{\partial}{\partial Fo}\int_0^{\delta} \Theta \, dy_1 + \frac{\partial}{\partial x}\int_0^{\delta}\left(\int_{y_1}^{\delta} \Theta \, dy\right)^{\frac{n+1}{n}} dy_1 = \left.\frac{\partial\Theta}{\partial y_1}\right|_0^{\delta}. \quad (15)$$

For $\Theta(Fo, x, y_1)$ we have

$$\frac{\partial^2\Theta}{\partial y_1^2} = 0 \quad \text{at} \quad y = 0. \quad (16)$$

Approximating the temperature profile by the third-power polynomial and satisfying (4), (13), (16), we arrive at

$$\Theta = 1 - \frac{3}{2}\frac{y_1}{\delta} + \frac{1}{2}\left(\frac{y_1}{\delta}\right)^3. \quad (17)$$

Substitution of (17) into the integral relationship yields upon transformations

$$\frac{\delta}{4}\frac{\partial\delta}{\partial Fo} + \frac{2(2n+1)}{8^{\frac{n+1}{n}}\,3n} C(n)\delta^{\frac{2n+1}{n}}\frac{\partial\delta}{\partial x} = 1, \quad (18)$$

where

$$C(n) = \int_0^1 (3-8\eta+6\eta^2-\eta^4)^{\frac{n+1}{n}}\,\mathrm{d}\eta.$$

To close the problem, conditions with $Fo = 0$ and $x = 0$ are needed. They are

$$\delta(0,x) = 0, \quad \delta(Fo,0) = 0. \tag{19}$$

The system of characteristics for (18) is written in the form

$$\frac{\mathrm{d}Fo}{\frac{\delta}{4}} = \frac{\mathrm{d}x}{\frac{2(2n+1)C(n)}{8^{\frac{n+1}{n}}3n}\delta^{\frac{2n+1}{n}}} = \mathrm{d}\delta. \tag{20}$$

This set of equations breaks down into the following equations

$$\mathrm{d}Fo = \frac{\delta}{4}\mathrm{d}\delta,$$

$$\mathrm{d}x = [2(2n+1)C(n)/(8^{\frac{n+1}{n}}3n)]\delta^{\frac{2n+1}{n}}\,\mathrm{d}\delta, \tag{21}$$

$$[2(2n+1)C(n)/(8^{\frac{n+1}{n}}3n)]\delta^{\frac{2n+1}{n}}\,\mathrm{d}Fo = \frac{\delta}{4}\mathrm{d}x.$$

Integration of the first equation in (12) with regard for (19) yields

$$\delta = 2\sqrt{(2Fo)} \qquad (Fo \leqslant Fo_s). \tag{22}$$

After integration of the second equation of system (21) with regard for (19) we obtain

$$\delta = \left[\frac{3(3n+1)8^{\frac{n+1}{n}}}{2(2n+1)C(n)}\right]^{\frac{n}{3n+1}} x^{\frac{n}{3n+1}} \qquad (Fo > Fo_s). \tag{23}$$

Relation (22) is valid for unsteady-state free convection regime, and equation (23) gives the relation for a steady-state free convection regime. The relation for limiting characteristics which determines time Fo_s necessary that a steady state be achieved at any x, is found from the third equation of system (21) allowing for equations (22) and (23)

$$Fo_s = \frac{1}{8}\left[\frac{3(3n+1)8^{\frac{n+1}{n}}}{2(2n+1)C(n)}\right]^{\frac{2n}{3n+1}} x^{\frac{2n}{3n+1}}.$$

The velocity of steady-state front propagation is

$$U = 8^{\frac{n-1}{n}}\frac{2n+1}{3n}C(n)Fo^{\frac{n+1}{2n}}.$$

The time of development of a steady state vs n and x (limiting characteristics) is plotted in Fig. 3. Increase in pseudoplasticity decreases sharply the time of development of a steady state at free convection, thus increasing heat fluxes on the surface.

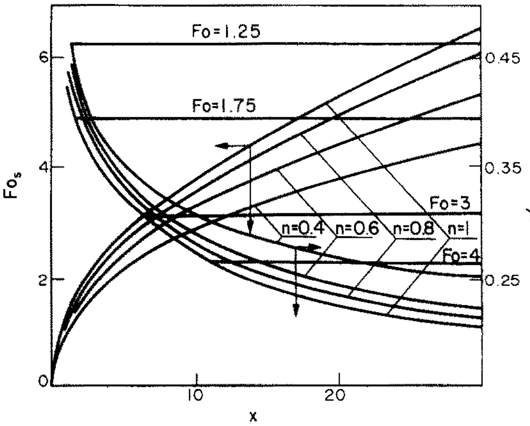

FIG. 3. Limiting and heat- and mass-transfer characteristics for unsteady free convection.

The local Nusselt number is

$$Nu = \frac{3}{2\delta}Gr^{\frac{1}{2(n+1)}}Pr^{\frac{n}{3n+1}}.$$

It should be concluded that the method of coupled asymptotic expansions may be applied not only when the wall temperature but also a heat flux are prescribed [5]. Besides, this statement allows the characteristics of heat and mass transfer and friction at a wall for thin bodies of revolution [6, 7] to be found.

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LA PREVISION DE LA CONVECTION NATURELLE DANS LES FLUIDES NON-NEWTONIENS

Résumé—On présente une méthode de prévision de la convection libre dans les fluides newtoniens et visqueux à comportement non-linéaire, aux nombres de Prandtl et de Schmidt élevés, sous forme de développements asymptotiques externes et internes couplés. On montre que la méthode permet essentiellement de simplifier la résolution du problème initial en fournissant en particulier des solutions de similitude. Des solutions sont présentées pour la convection naturelle stationnaire et non-stationnaire.

EIN VERFAHREN ZUR BERECHNUNG DER FREIEN KONVEKTION IN NICHT-NEWTONSCHEN FLUIDEN

Zusammenfassung—Es wird der Versuch unternommen, die freie Konvektion in nicht-Newtonschen und Newtonschen Fluiden bei hohen Prandtl- und Schmidt-Zahlen mit Hilfe gekoppelter, externer und interner asymptotischer Entwicklungen darzustellen. Es wird gezeigt, daß diese Methode die Lösung des ursprünglichen Problems wesentlich vereinfacht und speziell Ähnlichkeitslösungen erlaubt. Für stationäre und instationäre freie Konvektion werden Lösungen angegeben.

МЕТОД РЕШЕНИЯ ЗАДАЧ СВОБОДНОЙ КОНВЕКЦИИ В НЕНЬЮТОНОВСКИХ ЖИДКОСТЯХ

Аннотация — Представлен метод получения решения задач свободной конвекции нелинейновязких и ньютоновских жидкостей, имеющих большие числа Прандтля и Шмидта, а виде сращиваемых внешних и внутренних асимптотических разложений. Показано, что данный метод существенно упрощает решение исходной задачи, позволяя, в частности, переходить к автомодельным решениям. Найдены решения задачи как стационарной, так и нестационарной свободной конвекции.